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The Dynamical Behaviour of Test Particles in a Quasi-spherical Spacetime and the Physical Meaning of Superenergy

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Abstract We calculate the instantaneous proper radial acceleration of test particles (as measured by a locally defined Lorentzian observer) in a Weyl spacetime, close to the horizon. As expected from the Israel theorem, there appear some bifurcations with respect to the spherically symmetric case (Schwarzschild) which are explained in terms of the behaviour of the superenergy, bringing out the physical relevance of this quantity in the study of general relativistic systems.

1 Introduction

As it is well known, since the seminal paper by Israel [1], the only static and asymptotically-flat vacuum space-time possessing a regular horizon is the Schwarzschild solution. All the other Weyl exterior solutions [2–6], exhibit singularities in the physical components of the Riemann tensor at $r = 2M$.

For not particularly intense gravitational fields and small fluctuations off spherical symmetry, deviations from spherical symmetry may be described as perturbations of the spherically symmetric exact solution [7].

However, such perturbative scheme will eventually fail in regions close to the horizon (although strictly speaking the term "horizon" refers to the spherically symmetric case, we shall use it when considering the $r = 2M$ surface, in the case of small deviations from sphericity).

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Indeed, as we approach the horizon, any finite perturbation of the Schwarzschild spacetime becomes fundamentally different from the corresponding exact solution representing the quasi-spherical spacetime, even if the latter is characterized by parameters whose values are arbitrarily close to those corresponding to Schwarzschild metric [8–11]. This, of course, is just an expression of the Israel theorem (for observational differences between black holes and naked singularities see [12, 13] and references therein).

Therefore, for strong gravitational fields, no matter how small the multipole moments of the source are (those higher than monopole), there exists a bifurcation between the perturbed Schwarzschild metric and all the other Weyl metrics (in the case of gravitational perturbations).

Examples of such a bifurcation have been brought out in the study of the trajectories of test particles in the γ spacetime [14–21], and in the M-Q spacetime [22, 23], for orbits close to $2M$ [24, 25].

Also, the influence of the quadrupole moment on the motion of test particles within the context of Erez-Rosen metric [26] has been investigated by many authors (see [27–30] and references therein).

The purpose of this paper is to explain the bifurcation mentioned above, in terms of the behaviour of super-energy [31, 32] in a neighborhood of the horizon. This quantity, which may be defined from the Bel [33] or the Bel-Robinson tensor [34] (they both coincide in vacuum), has been shown to be very useful when it comes to explaining a number of phenomena in the context of general relativity.

Thus, for instance, it helps to explain the occurrence of vorticity in both radiative [35], and stationary spacetimes [36]. Also, it renders intelligible the behaviour of test particles moving in circles around the symmetry axis in an Einstein-Rosen spacetime [37].

In this paper we shall see how the behaviour of the instantaneous radial acceleration of a test particle (as measured by a locally defined Lorentzian observer) in a specific spacetime of the Weyl family and in regions close to the horizon, becomes intelligible when contrasted with the corresponding behaviour of superenergy.

The Weyl metric to be considered here is the M-Q spacetime. The rationale for this choice is that due to its relativistic multipole structure, the M-Q solution (more exactly, a sub-class of this solution M-Q$^{(1)}$, [22]) may be interpreted as a quadrupole correction to the Schwarzschild space-time, and therefore represents a good candidate among known Weyl solutions, to describe small deviations from spherical symmetry.

For this metric we shall calculate the instantaneous radial acceleration of a test particle and the superenergy. Then the very peculiar behaviour of the former (close to the horizon) will be explained in terms of the behaviour of the latter.

The paper is structured as follows: in Sect. 2 we present the Weyl family of metrics we shall be concerned with and briefly discuss some of its properties; next, in Sect. 3, we calculate the radial four-acceleration of test particles in such setup. In Sect. 4 we review and discuss the concept of superenergy. In Sect. 5 we briefly describe the M-Q solution and particularize the expressions for the proper radial four-acceleration and the superenergy for the case of the M-Q$^{(1)}$ metric. Finally results are discussed in last section.

2 The Weyl Metrics

Static axisymmetric solutions to Einstein's equations are given by the Weyl metric [2–6]

$$ds^2 = e^{2\Psi} dt^2 - e^{-2\Psi}[e^{2\Gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2]. \tag{1}$$

For vacuum spacetimes, Einstein's Field Equations imply for the metric functions

$$\Psi_{,\rho\rho} + \rho^{-1}\Psi_{,\rho} + \Psi_{,zz} = 0 \tag{2}$$

and

$$\Gamma_{,\rho} = \rho(\Psi_{,\rho}^2 - \Psi_{,z}^2); \qquad \Gamma_{,z} = 2\rho\Psi_{,\rho}\Psi_{,z}. \tag{3}$$

Notice that (2) is just the Laplace equation for Ψ (in 2-dimensional Euclidean space); furthermore, it is precisely the integrability condition for (3), that is: given Ψ, a function Γ satisfying (3) always exists. Since in the weak field limit Ψ is related to the Newtonian gravitational potential, this result may be stated as saying that for any "Newtonian" potential there always exists a specific Weyl metric, a well known result.

An interesting way of writing the general solution of (2, 3) was obtained by Erez-Rosen [26] and Quevedo [38], using prolate spheroidal coordinates, which are defined as follows

$$\begin{aligned} x &= \frac{r_+ + r_-}{2\sigma}, \qquad y = \frac{r_+ - r_-}{2\sigma}, \\ r_\pm &\equiv [\rho^2 + (z \pm \sigma)^2]^{1/2}, \\ x &\geq 1, \qquad -1 \leq y \leq 1, \end{aligned} \tag{4}$$

where σ is an arbitrary constant which will be identified later with the Schwarzschild's mass. The prolate coordinate x represents a radial coordinate, whereas the other coordinate, y represents the cosine function of the polar angle.

In these prolate spheroidal coordinates, Ψ takes the form

$$\Psi = \sum_{n=0}^{\infty} (-1)^{n+1} q_n Q_n(x) P_n(y), \tag{5}$$

where $P_n(x)$ and $Q_n(y)$ are the Legendre functions of first and second kind respectively, and q_n a set of arbitrary constants. The corresponding expression for the function Γ, may be found in [38].

3 The Radial Acceleration of Test Particles

In order to find an expression for the instantaneous radial acceleration of test particles, it is useful to start from the geodesic equations.

These can be derived from the Lagrangian

$$2\mathcal{L} = g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta, \tag{6}$$

where the dot denotes differentiation with respect to an affine parameter s, which for time-like geodesics coincides with the proper time. Then, from Euler-Lagrange equations it follows,

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0, \tag{7}$$

we shall not need the full set of geodesic equations, therefore we shall display only the one involving radial acceleration

$$2\ddot{r} g_{rr} + 2\dot{r}(\dot{r} g_{rr,r} + g_{rr,\theta}\dot{\theta}) - \dot{t}^2 g_{tt,r} - \dot{r}^2 g_{rr,r} - \dot{\theta}^2 g_{\theta\theta,r} - \dot{\phi}^2 g_{\phi\phi,r} = 0, \tag{8}$$

where, instead of cylindrical coordinates (ρ, z), we found useful to work with Erez-Rosen coordinates (r, θ) given by:

$$\begin{aligned} z &= (r - M)\cos\theta, \\ \rho &= (r^2 - 2Mr)^{1/2}\sin\theta \end{aligned} \tag{9}$$

which are related to prolate coordinates, by

$$\begin{aligned} x &= \frac{r}{M} - 1, \\ y &= \cos\theta. \end{aligned} \tag{10}$$

Since we are concerned only with timelike geodesics, the range of our coordinates is:

$$\infty > t \geq 0 \qquad r > 2M \qquad \pi \geq \theta \geq 0 \qquad 2\pi \geq \phi \geq 0.$$

Let us now consider the motion of a test particle along a radial geodesic, for an arbitrary value of θ. Thus putting $\dot{\theta} = \dot{\phi} = 0$ in (8), and using the constraint (for radial geodesics)

$$1 = g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2, \tag{11}$$

we obtain

$$2\ddot{r}g_{tt}g_{rr} + \dot{r}^2(g_{rr}g_{tt})_{,r} - g_{tt,r} = 0. \tag{12}$$

It should be kept in mind that we are not interested in a full description of the motion of test particles (we are not going to integrate the full set of geodesic equations), but only in the expression for the radial acceleration of a test particle at any given time. Accordingly we shall not need to take into consideration the constraints imposed on $\dot{r}$, $\ddot{\theta}$ and so on, from the other geodesic equations, for the radial motion.

In order to express our results in terms of physically meaningful quantities, let us introduce (locally defined) coordinates associated with a locally Minkowskian observer (or alternatively, a tetrad field associated with such a Minkowskian observer). Thus, let

$$dX = \sqrt{-g_{rr}}dr \tag{13}$$

and

$$dT = \sqrt{g_{tt}}dt. \tag{14}$$

It then follows that

$$\dot{r} = \frac{\frac{dX}{dT}}{\sqrt{g_{rr}[(\frac{dX}{dT})^2 - 1]}} \tag{15}$$

and

$$\frac{d^2X}{dT^2} = \ddot{r}\sqrt{-g_{rr}}\left[1 - \left(\frac{dX}{dT}\right)^2\right]^2 - \left(\frac{dX}{dT}\right)^2 \frac{g_{rr,r}[1 - (\frac{dX}{dT})^2]}{2(-g_{rr})^{3/2}}. \tag{16}$$

In the spherically symmetric case, (16) reduces to

$$\frac{d^2X}{dT^2} = -\frac{M}{r^2}\left[1 - \left(\frac{dX}{dT}\right)^2\right]\left(1 - \frac{2M}{r}\right)^{-1/2}, \tag{17}$$

or, introducing the variable $R = \frac{r}{M}$

$$\frac{d^2X}{dT^2} = -\frac{1}{MR^{3/2}}\left[1 - \left(\frac{dX}{dT}\right)^2\right](R - 2)^{-1/2}, \tag{18}$$

which is a known result. Since dX/dT is always smaller than one, the attractive nature of gravity for any value of r (larger than $2M$) is clearly exhibited in (17).

4 Superenergy

As it is known, in classical field theory, energy is a quantity defined in terms of potentials and their first derivatives. In General Relativity however, it is impossible to construct a tensor expressed only through the metric and their first derivatives (the equivalence principle). Accordingly, a local description of gravitational energy in terms of true invariants (tensors of any rank) is not possible within the context of the theory.

Thus, one is left with the following three alternatives:

- Looking for a non-local definition of energy
- Finding a definition based on pseudo-tensors
- Resorting to a succedaneous definition, e.g.: superenergy.

In this work we are going to explore the last alternative. As indicated in the Introduction, the motivations for doing so are given by the rich and profound physical meaning of such quantity.

Superenergy W may be defined from either the Bel or the Bel-Robinson tensor [39]. Since we are working with vacuum spacetimes both definitions coincide, and one then has:

$$W = E^{\alpha\beta}E_{\alpha\beta} + B^{\alpha\beta}B_{\alpha\beta} \tag{19}$$

with

$$E_{\alpha\beta} = C_{\alpha\gamma\beta\delta}u^{\gamma}u^{\delta}, \tag{20}$$

$$B_{\alpha\beta} =^{*} C_{\alpha\gamma\beta\delta}u^{\gamma}u^{\delta} = \frac{1}{2}\eta_{\alpha\gamma\epsilon\rho}C^{\epsilon\rho}{}_{\beta\delta}u^{\gamma}u^{\delta}, \tag{21}$$

where $C_{\alpha\gamma\beta\delta}$ is the Weyl tensor, $\eta_{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor and u^{α} is the four-velocity of observers at rest in the frame of (1), i.e.

$$u^{\alpha} = \left(\frac{1}{\sqrt{g_{00}}}, 0, 0, 0\right). \tag{22}$$

Observe that since we are working with static spacetimes, the magnetic part of the Weyl tensor ($B_{\alpha\beta}$) vanishes identically.

Let us next briefly introduce the metric we shall consider here, and calculate the corresponding expressions for the radial acceleration of a test particle and the superenergy.

5 The Monopole-Quadrupole Solution, *M*-*Q*

In [22, 23] it was shown that it is possible to find a metric of the Weyl family, such that the resulting solution possesses only monopole and quadrupole moments (in the Geroch sense [40–42]). The obtained solution (M-Q) may be written as follows:

$$\Psi_{M\text{-}Q} = \Psi_{q^0} + q\Psi_{q^1} + q^2\Psi_{q^2} + \cdots = \sum_{\alpha=0}^{\infty} q^{\alpha}\Psi_{q^{\alpha}}, \tag{23}$$

where the zeroth order corresponds to the Schwarzschild solution.

$$\Psi_{q^0} = -\sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{2n+1} P_{2n}(\cos\theta), \tag{24}$$

with $\lambda \equiv M/r$ and it appears that each power in q adds a quadrupole correction to the spherically symmetric solution. Now, it should be observed that due to the linearity of Laplace equation, these corrections give rise to a series of exact solutions. In other words, the power series of q may be cut at any order, and the partial summation, up to that order, gives an exact solution representing a quadrupolar correction to the Schwarzschild solution.

Since we are interested in slight deviations from spherical symmetry, we shall consider the M-Q solution, only up to the first order in q (M-Q$^{(1)}$); with $q > 0$ ($q < 0$) corresponding to an oblate (prolate) source.

Thus, the explicit solution up to the first order, describing a quadrupolar correction to the monopole (Schwarzschild solution), may be interpreted as the gravitational field outside a quasi-spherical source, and it is given by (note a misprint in (13) in [23])

$$\begin{aligned}
\Psi^{(1)}_{M\text{-}Q} \equiv \Psi_{q^0} + q\Psi_{q^1} = {} & \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + \frac{5}{8}q(3y^2-1) \\
& \times \left[\left(\frac{3x^2-1}{4} - \frac{1}{3y^2-1}\right)\ln\left(\frac{x-1}{x+1}\right)\right. \\
& \left. - \frac{2x}{(x^2-y^2)(3y^2-1)} + \frac{3x}{2}\right],
\end{aligned} \tag{25}$$

$$\begin{aligned}
\Gamma^{(1)}_{M\text{-}Q} \equiv {} & \Gamma_{q^0} + q\Gamma_{q^1} + q^2\Gamma_{q^2} = \frac{1}{2}\left(1 + \frac{225}{24}q^2\right)\ln\left(\frac{x^2-1}{x^2-y^2}\right) \\
& - \frac{15}{8}qx(1-y^2)\left[1 - \frac{15}{32}\left(x^2 + 7y^2 - 9x^2y^2 + 1\right.\right. \\
& \left.\left. - \frac{8}{3}\frac{x^2+1}{x^2-y^2}\right)\right]\ln\left(\frac{x-1}{x+1}\right) \\
& + \frac{225}{1024}q^2(x^2-1)(1-y^2)(x^2+y^2-9x^2y^2-1)\ln^2\left(\frac{x-1}{x+1}\right) \\
& - \frac{15}{4}q(1-y^2)\left[1 - \frac{15}{64}q(x^2+4y^2-9x^2y^2+4)\right] \\
& - \frac{75}{16}q^2x^2\frac{1-y^2}{x^2-y^2} - \frac{5}{4}q(x^2+y^2)\frac{1-y^2}{(x^2-y^2)^2} \\
& - \frac{75}{192}q^2(2x^6 - x^4 + 3x^4y^2 - 6x^2y^2 \\
& + 4x^2y^4 - y^4 - y^6)\frac{1-y^2}{(x^2-y^2)^4}.
\end{aligned} \tag{26}$$

In [25] it was shown that the behaviour of test particles in the M-Q$^{(1)}$ spacetime, becomes particularly strange on the symmetry axis; i.e.: $\theta = 0, \pi$, or else $y = \pm 1$ (close to the horizon). Therefore it is for that region that we are going to calculate the proper radial acceleration of a particle on the axis for the M-Q$^{(1)}$ spacetime (for all other regions, including the equatorial plane, the abnormal behaviour commented below is not observed [25]). Using (16) we obtain

$$\begin{aligned}\frac{d^2X(T)}{dT^2} = {} & \frac{1}{8}e^{(5/4)qA(R)}\left(\frac{dX(T)}{dT}+1\right)\left(\frac{dX(T)}{dT}-1\right)\sqrt{\frac{(R-2)^{-3}}{R^5}} \\ & \times\left[15q\ln\left(\frac{R-2}{R}\right)(R^5-5R^4+8R^3-4R^2)\right. \\ & + q(30R^4-120R^3+130R^2-20R+20) \\ & \left. + 8R^2-16R\right]\frac{1}{M}\end{aligned} \tag{27}$$

where

$$\begin{aligned}A(R) = {} & \frac{1}{R(R-2)}\left[6R^3-18R^2+8R+4\right. \\ & \left. + \ln\left(\frac{R-2}{R}\right)(3R^4-12R^3+12R^2)\right].\end{aligned} \tag{28}$$

Since we are interested in the value of the radial acceleration for Lorentzian observers instantaneous at rest, we shall plot (27) with $q = \pm 0.01$ and $\frac{dX(T)}{dT} = 0$.

For this metric, the expression of the superenergy (again, on the symmetry axis $y^2 = 1$) reads

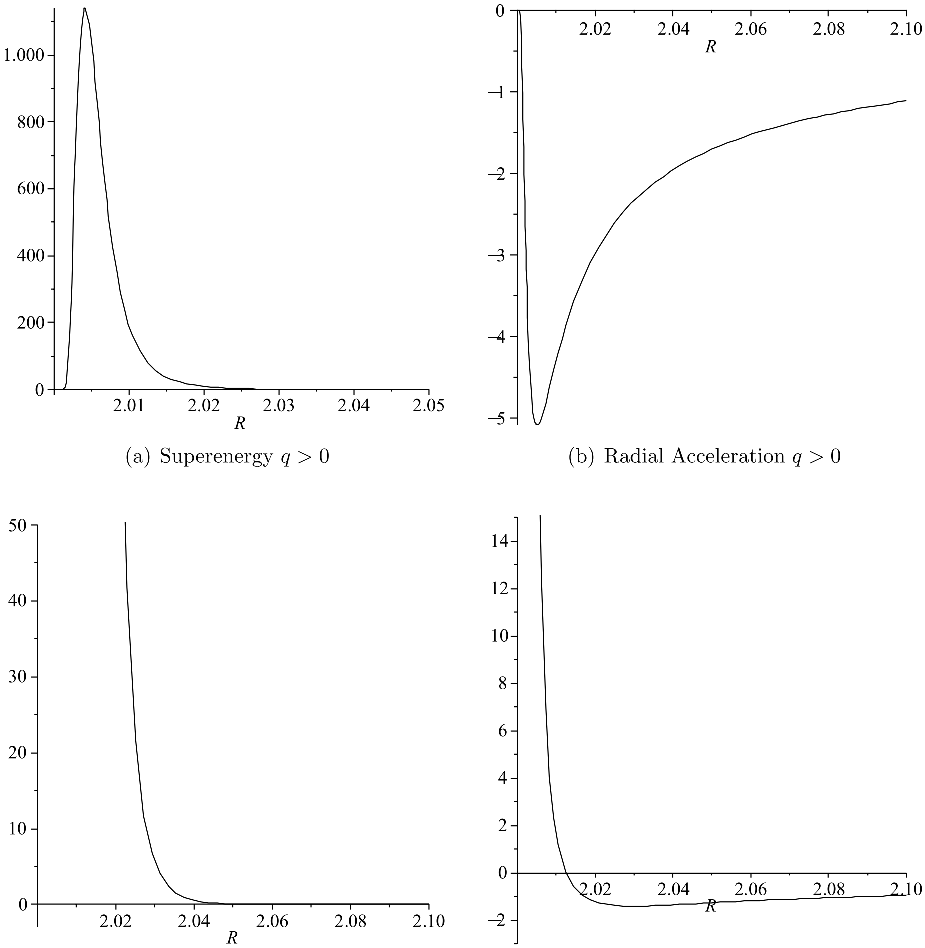
$$\begin{aligned}W_{MQ} = {} & \frac{1}{1536}e^{(5/4)qA(R)} \times \left[(768R^2-1152R^3+576R^4-96R^5)\right. \\ & + q(-1920R+3360R^2-6720R^3+5880R^4 \\ & - 1200R^5-540R^6+180R^7) \\ & + q^2(400-800R+5600R^2-10000R^3+22900R^4 \\ & - 32400R^5+22200R^6-7200R^7+900R^8) \\ & + q\ln\left(\frac{R-2}{R}\right)(2880R^3-5760R^4+3600R^5 \\ & - 360R^6-360R^7+90R^8) \\ & + q^2\ln\left(\frac{R-2}{R}\right)(-2400R^2+7200R^3-23400R^4+49200R^5 \\ & - 52500R^6+29100R^7-8100R^8+900R^9) \\ & + q^2\ln\left(\frac{R-2}{R}\right)^2(3600R^4-14400R^5+23400R^6 \\ & \left. - 19800R^7+9225R^8-2250R^9+225R^{10})\right] \\ & \times \frac{1}{M^4(R-2)^6R^{10}}.\end{aligned} \tag{29}$$

6 Discussion

The first conclusion which emerges from Figs. 1(a) and (b) is that, close to the horizon, the behaviour of the test particle (in what concerns $d^2X(T)/dT^2$) is extremely abnormal, as expected from Israel's theorem.

Thus for the M-Q$^{(1)}$ metric with $q > 0$ it appears that a test particle placed on the axis of symmetry in the neighbourhood of the horizon does not feel any attraction from the source ($d^2X(T)/dT^2 \approx 0$). Still more shocking: as we move outwards (always on the symmetry axis), the magnitude of $\frac{d^2X(T)}{dT^2}$ increases with R, until some value of R, from which it starts to "behave" properly (Fig. 1(b)).

This pathological behaviour of $d^2X(T)/dT^2$ is fully consistent with that of superenergy in the same range of R, as indicated in Fig. 1(a). Indeed, W also vanishes close to the



(a) Superenergy $q > 0$

(b) Radial Acceleration $q > 0$

(c) Superenergy $q < 0$

(d) Radial Acceleration $q < 0$

Fig. 1 (**a**) and (**b**) show the behaviour of W and $\frac{d^2X(T)}{dT^2}$ for $q = 0.01$, whereas (**c**) and (**d**) display their behaviour for $q = -0.01$

horizon, increasing as we move outwards along the symmetry axis, until we are far away enough from the horizon and the expected behaviour is recovered.

For $q < 0$ the situation is still more unusual. Indeed, on a neighbourhood of the horizon, on the axis of symmetry, $d^2X(T)/dT^2 > 0$, implying that the particle experiences a repulsive force. This effect is restricted to values of R very close to 2. As we move away from the horizon the proper acceleration becomes negative, although still displaying an abnormal behaviour since it increases in magnitude with R. Moving further away from the origin (along the symmetry axis) we recover the "normal" behaviour ($d^2X(T)/dT^2$ (negative and decreasing with R). The dependence of W with R in this case, displayed in Fig. 1(c), is consistent with the graphics of $d^2X(T)/dT^2$ above. Indeed, in a neighborhood of the horizon, W is singular and so is its derivative with respect to R, this explaining the pathological behaviour of $d^2X(T)/dT^2$ in that range of values of R. As we move sufficiently far away from $R = 2$ we recover the expected behaviour.

Thus we have seen that the concept of superenergy is a suitable measure of the strength of gravitational interaction, even in highly pathological situations. The fact that it is a true scalar (obtained from a true tensor) reinforces further its relevance in the study of self–gravitating systems.

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